

A flame propagation model on a network with application to a blocking problem

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Abstract

We consider the Cauchy problem

$$\begin{cases} \partial_t u + H(x, Du) = 0 & (x, t) \in \Gamma \times (0, T) \\ u(x, 0) = u_0(x) & x \in \Gamma \end{cases}$$

where Γ is a network and H is a convex and positive homogeneous Hamiltonian. We introduce a definition of viscosity solution and we prove that the unique viscosity solution of the problem is given by a Hopf-Lax type formula. In the second part of the paper we study flame propagation in a network and we seek an optimal strategy to block a fire breaking up in some part of a pipeline.

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1 Introduction

We study the Cauchy problem

$$\begin{cases} \partial_t u + H(x, Du) = 0 & (x, t) \in \Gamma \times (0, T) \\ u(x, 0) = u_0(x) & x \in \Gamma \end{cases} \quad (1.1)$$

where Γ is a network and the operator $H : \Gamma \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, convex, nonnegative and positive homogeneous in the last variable.

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In \mathbb{R}^n , the problem (1.1) arises in flame propagation models and evolution of curves whose speed of propagation only depends on the normal direction. Existence, uniqueness and evolution of level sets of the solution of (1.1) have been extensively studied in the framework of viscosity solution theory (see [3, 4, 17]). The unique viscosity solution of (1.1) is given by the Hopf-Lax formula

$$u(x, t) = \min\{u_0(y) : S(y, x) \leq t\} \quad (x, t) \in \mathbb{R}^n \times (0, \infty) \quad (1.2)$$

where S is a distance function characterized by solving the associated stationary equation

$$H(x, Du) = 1 \quad x \in \mathbb{R}^n.$$

In the recent time, there is an increasing interest in the study of nonlinear differential equations on networks since they describe various phenomena as traffic flow, blood circulation, data transmission, electric networks, etc (see [9, 12]). Concerning Hamilton-Jacobi equations on networks, we mention the recent papers [1, 6, 8, 11, 14].

In this paper, following the approach in [14], we cope existence, uniqueness and regularity for evolutive Hamilton-Jacobi equations on networks. In particular we prove that the Hopf-Lax formula (1.2) can be extended to this framework.

The main issue of our investigation is to tackle transition vertices (namely, points of the network where several edges meet each other). Actually, in our framework a suitable definition of viscosity solution at transition vertices (together with the standard one at points inside edges) will ensure the well posedness of the problem. Let us recall that this feature also happens for stationary first order equations (see [1, 11, 14]) whereas, for second order equations, some transition conditions (the so-called Kirchhoff condition) need to be imposed (see [12] and references therein).

In the second part of the paper we illustrate our results with a concrete application: the *blocking* problem. Suppose that a fire breaks up in some region of an oil pipeline. A central controller can stop the propagation of the fire by closing the junctions of the pipes, represented by the vertices of the network. The controller spends some time to reach the junctions which become unavailable when they are reached by the fire front. Therefore only a subset of the vertices can be closed on time to get fire under control. The aim is to find a strategy which maximizes the part of the network preserved by the fire. We give a characterization of the optimal strategy and we study the corresponding flame propagation in the network. Moreover we describe a numerical scheme for the solution of the problem and we present some numerical examples.

Notations: A network Γ is a finite collection of points $V := \{x_i\}_{i \in I}$ and edges $E := \{e_j\}_{j \in J}$ in \mathbb{R}^n . The vertices of V are connected by the continuous, non self-intersecting arcs of E . Each arc e_j is parametrized by a smooth function $\pi_j : [0, l_j] \rightarrow \mathbb{R}^n$, $l_j > 0$.

For $i \in I$ we set $Inc_i := \{j \in J \mid e_j \text{ is incident to } x_i\}$. We set $I_B := \{i \in I \mid \#Inc_i = 1\}$, $I_T := I \setminus I_B$, and we denote by $\partial\Gamma := \{x_i \in V \mid i \in I_B\}$, the set of boundary vertices of Γ , and by $\Gamma_T := \{x_i \mid i \in I_T\}$, the set of transition vertices. For

simplicity, we assume $\partial\Gamma = \emptyset$ (otherwise, one can introduce appropriate boundary condition on $\partial\Gamma$).

The network is not oriented, but the parametrization of the arcs induces an orientation which can be expressed by the *signed incidence matrix* $A = \{a_{ij}\}_{i \in I, j \in J}$ with

$$a_{ij} := \begin{cases} 1 & \text{if } x_i \in e_j \text{ and } \pi_j(0) = x_i, \\ -1 & \text{if } x_i \in e_j \text{ and } \pi_j(l_j) = x_i, \\ 0 & \text{otherwise.} \end{cases}$$

In the following we always identify $x \in e_j$ with $y = \pi_j^{-1}(x) \in [0, l_j]$. For any function $u : \Gamma \rightarrow \mathbb{R}$ and each $j \in J$ we denote by $u_j : [0, l_j] \rightarrow \mathbb{R}$ the restriction of u to e_j , i.e. $u_j(y) = u(\pi_j(y))$ for $y \in [0, l_j]$. The derivative are always considered with respect to the parametrization of the arc, i.e. if $x \in e_j$, $y = \pi_j^{-1}(x)$ then $Du(x) := \frac{du_j}{dy}(y)$. At $x = x_i \in V$, we write by $D_j u(x)$ to intend the derivative relative to the arc e_j , $j \in \text{Inc}_i$.

We denote by $d : \Gamma \times \Gamma \rightarrow \mathbb{R}^+$ the path distance on Γ , i.e.

$$d(x, y) := \inf \{ \ell(\gamma) : \gamma \subset \Gamma \text{ is a path joining } x \text{ to } y \}, \quad x, y \in \Gamma. \quad (1.3)$$

2 Evolutive Hamilton-Jacobi equations on networks

In this section we consider the Hamilton-Jacobi equation

$$\partial_t u + H(x, Du) = 0 \quad (x, t) \in \Gamma \times (0, T) \quad (2.1)$$

with the initial condition

$$u(x, 0) = u_0(x_0) \quad x \in \Gamma. \quad (2.2)$$

The Hamiltonian $H : \Gamma \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following assumptions

$$H \in C^0(\Gamma \times [0, T)); \quad (2.3)$$

$$|H(x, p) - H(y, p)| \leq Cd(x, y)(1 + |p|) \quad \text{for any } x, y \in \Gamma, p \in \mathbb{R}; \quad (2.4)$$

$$H(x, p) = H(x, -p) \quad \text{for any } x = x_i \in V, p \in \mathbb{R}; \quad (2.5)$$

$$H(x, \cdot) \text{ is convex and positive homogeneous in } p \text{ for any } x \in \Gamma; \quad (2.6)$$

$$\inf \{ H(x, p) : |p| = 1, x \in \Gamma \} > 0. \quad (2.7)$$

Remark 2.1 (2.3)–(2.4) are standard assumptions in the framework of viscosity solution theory. Assumption (2.5) is the independence of H of the parametrization of the arc: if we change the parametrization of e_j incident to x_i and we invert its orientation then also the derivative $\frac{du_j}{dy}(y)$ for $y = \pi_j^{-1}(x_i)$ changes sign. By (2.6), the equation is called *geometric* and it is connected with front propagation (see [3, 4, 17]), while (2.7) implies in particular the coercivity of the Hamiltonian, i.e. $\lim_{|p| \rightarrow \infty} H(x, p) = +\infty$ for any $x \in \Gamma$.

Example 1 An Hamiltonian satisfying the previous assumptions is given by

$$H(x, p) = \sup_{a \in A} \{-b(x, a)p\}$$

where A is a compact metric space, $b : \Gamma \times A \rightarrow \mathbb{R}$ is a continuous function such that, for some $r > 0$, there holds $(-r, r) \subset \overline{\text{co}}\{b(x, a) : a \in A\}$ and $\sup\{b(x, a) : a \in A\} = \sup\{-b(x, a) : a \in A\}$. In particular, if $A = (-1, 1)$ and $b(x, a) = a/c(x)$ with c bounded and strictly positive, then the Hamiltonian is $H(x, p) = |p|/c(x)$ (called eikonal Hamiltonian) and it satisfies assumptions (2.3)-(2.7).

In the next definitions we introduce the class of test functions and the notion of viscosity solution for (2.1).

Definition 2.1 Let $\phi \in C(\Gamma \times (0, T))$.

- i) Let $(x, t) \in e_j \times (0, T)$, $j \in J$. We say that ϕ is test function at (x, t) , if ϕ_j is differentiable at $(\pi_j^{-1}(x), t)$.
- ii) Let $(x, t) = (x_i, t)$, $i \in I_T$, $j, k \in \text{Inc}_i$, $j \neq k$, $t \in (0, T)$. We say that ϕ is (j, k) -test function at (x, t) , if ϕ_j and ϕ_k are differentiable respectively at $(\pi_j^{-1}(x), t)$ and $(\pi_k^{-1}(x), t)$, with

$$a_{ij}D_j\phi(\pi_j^{-1}(x), t) + a_{ik}D_k\phi(\pi_k^{-1}(x), t) = 0, \quad (2.8)$$

where (a_{ij}) is the signed incidence matrix.

Remark 2.2 By the continuity of ϕ and point (ii) we infer

$$\partial_t\phi_j(\pi_j^{-1}(x), t) = \partial_t\phi_k(\pi_k^{-1}(x), t).$$

Definition 2.2

- i) If $(x, t) \in e_j \times (0, T)$, $j \in J$, then a function $u \in USC(\Gamma \times (0, T))$ (resp., $v \in LSC(\Gamma \times (0, T))$) is called a subsolution (resp. supersolution) of (2.1) at (x, t) if for any test function ϕ for which $u - \phi$ attains a local maximum (resp., a local minimum) at (x, t) , we have

$$\partial_t\phi(x, t) + H(x, D\phi(x, t)) \leq 0 \quad (2.9)$$

$$(\text{resp., } \partial_t\phi(x, t) + H(x, D\phi(x, t)) \geq 0). \quad (2.10)$$

- ii) If $(x, t) = (x_i, t)$, $i \in I_T$, $t \in (0, T)$ then

- A function $u \in USC(\Gamma \times (0, T))$ is called a subsolution at (x, t) if for any $j, k \in \text{Inc}_i$ and any (j, k) -test function ϕ for which $u - \phi$ attains a local maximum at (x, t) relatively to $(e_j \cup e_k) \times (0, T)$, then

$$\partial_t\phi(x, t) + H(x, D_j\phi(x, t)) \leq 0. \quad (2.11)$$

- A function $v \in LSC(\Gamma \times (0, T))$ is called a *supersolution* at (x, t) if for any $j \in Inc_i$, there exists $k \in Inc_i$, $k \neq j$ (said *i-feasible* for j at (x, t)) such that for any (j, k) -test function ϕ for which $u - \phi$ attains a local minimum at (x, t) relatively to $(e_j \cup e_k) \times (0, T)$, we have

$$\partial_t \phi(x, t) + H(x, D_j \phi(x, t)) \leq 0. \quad (2.12)$$

Remark 2.3 By Remark 2.2 and assumption (2.5), at a vertex $x = x_i$, it holds

$$\partial_t \phi(x, t) + H(x, D_j \phi(x, t)) = \partial_t \phi(x, t) + H(x, D_k \phi(x, t))$$

In the next proposition we exploit the geometric character of (2.1) to give equivalent definitions of subsolution and supersolution.

Proposition 2.1 (i) A function $u \in USC(\Gamma \times (0, T))$ is a *subsolution* (respectively, $v \in LSC(\Gamma \times (0, T))$ is a *supersolution*) of (2.1) at $(x, t) \in e_j \times (0, T)$ if and only if for any $\alpha \in \mathbb{R}$ and for any admissible test function ϕ which has a local minimum on $\{u \geq \alpha\} \cap (e_j \times (0, T))$ (resp., a local maximum on $\{v \leq \alpha\} \cap (e_j \times (0, T))$) at (x, t) , then (2.9) (resp., (2.10)) holds.

(ii) A function $u \in USC(\Gamma \times (0, T))$ is a *subsolution* of (2.1) at $(x, t) = (x_i, t)$, $i \in I_T$, $t \in (0, T)$ if and only if for any $\alpha \in \mathbb{R}$, for any $j, k \in Inc_i$ with $i \neq j$, and any (j, k) -test function ϕ which has a local minimum on $\{u \geq \alpha\} \cap ((e_j \cup e_k) \times (0, T))$ at (x, t) , then (2.11) holds.

(iii) A function $v \in LSC(\Gamma \times (0, T))$ is a *supersolution* of (2.1) at $(x, t) = (x_i, t)$, $i \in I_T$, $t \in (0, T)$ if and only if for any $\alpha \in \mathbb{R}$, for any $j \in Inc_i$, there exists $k \in Inc_i$, $k \neq j$ such that for any (j, k) -test function ϕ for which $v - \phi$ has a local maximum on $\{v \leq \alpha\} \cap ((e_j \cup e_k) \times (0, T))$ at (x, t) then (2.12) holds.

The proof is postponed to the Appendix.

2.1 A comparison theorem

We prove a comparison principle for the equation (2.1). We shall use some techniques introduced for evolutive equations in the Euclidean setting (see [2] and references therein) and for stationary equations on networks (see [14, Theorem 5.1]). Note that for the next result we only exploit assumptions (2.3)–(2.5).

Theorem 2.1 Let $u \in USC(\Gamma \times [0, T])$, $v \in LSC(\Gamma \times [0, T])$ be a subsolution and, respectively, a supersolution of (2.1) such that $u(x, 0) \leq v(x, 0)$ for $x \in \Gamma$. Then $u \leq v$ in $\Gamma \times [0, T]$.

PROOF We proceed by contradiction, assuming

$$\delta_1 := \sup_{\Gamma \times [0, T]} (u - v) > 0.$$

For η sufficiently small, the function $u(x, t) - v(x, t) - \eta/(T - t)$ attains a positive maximum at some point $(\bar{x}, \bar{t}) \in \Gamma \times (0, T)$ and we set

$$\delta_0 := u(\bar{x}, \bar{t}) - v(\bar{x}, \bar{t}) - \eta/(T - \bar{t}) = \sup_{(x, t) \in \Gamma \times [0, T)} \left(u(x, t) - v(x, t) - \frac{\eta}{T - t} \right) > 0. \quad (2.13)$$

For each $\delta \in (0, \delta_0/2T)$, $\alpha \in (0, 1)$, we set

$$\Psi(x, t, \xi, s) := u(x, t) - v(\xi, s) - \delta t - \frac{d(x, \xi)^2}{2\alpha} - \frac{|t - s|^2}{2\alpha} - \frac{\eta}{T - t}$$

where d is the distance defined in (1.3). By the choice of δ and η , we have

$$\Psi(\bar{x}, \bar{t}, \bar{x}, \bar{t}) = u(\bar{x}, \bar{t}) - v(\bar{x}, \bar{t}) - \delta \bar{t} - \frac{\eta}{T - \bar{t}} = \delta_0 - \delta \bar{t} > \frac{\delta_0}{2};$$

whence, the function Ψ attains a positive maximum with respect to $\tilde{G} := \Gamma \times [0, T) \times \Gamma \times [0, T]$ at some point $(\tilde{x}, \tilde{t}, \tilde{\xi}, \tilde{s})$. Moreover, by the last relation, for every α , we have

$$\sup_{\tilde{G}} \Psi \geq \delta_0/2. \quad (2.14)$$

For later use, we denote by $(\bar{x}_1, \bar{t}_1) \in \Gamma \times [0, T)$ a maximum point of $\Psi(x, t, x, t)$ in $\Gamma \times [0, T)$, namely

$$\sup_{\Gamma \times [0, T)} \left(u(t, x) - v(t, x) - \frac{\eta}{T - t} - \delta t \right) = u(\bar{x}_1, \bar{t}_1) - v(\bar{x}_1, \bar{t}_1) - \frac{\eta}{T - \bar{t}_1} - \delta \bar{t}_1. \quad (2.15)$$

By the inequality $\Psi(\tilde{x}, \tilde{t}, \tilde{\xi}, \tilde{s}) \geq \Psi(\tilde{x}, \tilde{t}, \tilde{x}, \tilde{t})$ and the boundedness of Γ , we get

$$\frac{d(\tilde{x}, \tilde{\xi})^2}{2\alpha} + \frac{|\tilde{t} - \tilde{s}|^2}{2\alpha} \leq v(\tilde{x}, \tilde{t}) - v(\tilde{\xi}, \tilde{s}) \leq c$$

for some constant c independent of η , δ and α ; in particular, we infer the estimates

$$d(\tilde{x}, \tilde{\xi}) \leq c\sqrt{\alpha}, \quad |\tilde{t} - \tilde{s}| \leq c\sqrt{\alpha}.$$

It follows that, since $\tilde{t} < T$, $\tilde{s} < T$ for α sufficiently small.

We deduce that, as $\alpha \rightarrow 0$ (possibly passing to a subsequence), there holds

$$\tilde{x}, \tilde{\xi} \rightarrow x_0 \quad \text{and} \quad \tilde{t}, \tilde{s} \rightarrow t_0 \quad \text{as } \alpha \rightarrow 0^+$$

for some $(x_0, t_0) \in \Gamma \times [0, T)$. We claim that, in fact, there holds

$$\frac{d(\tilde{x}, \tilde{\xi})^2}{\alpha} + \frac{|\tilde{t} - \tilde{s}|^2}{\alpha} \rightarrow 0 \quad \text{as } \alpha \rightarrow 0. \quad (2.16)$$

Indeed, if the claim is false, by the semicontinuity of u and v , (recall that (\bar{x}_1, \bar{t}_1) is a maximum point for $\Psi(x, t, x, t)$ in $\Gamma \times [0, T)$) passing to the limit in the inequality

$$u(\tilde{x}, \tilde{t}) - v(\tilde{\xi}, \tilde{s}) - \frac{d(\tilde{x}, \tilde{\xi})^2}{2\alpha} - \frac{|\tilde{t} - \tilde{s}|^2}{2\alpha} - \frac{\eta}{T - \tilde{t}} - \delta\tilde{t} \geq u(\bar{x}_1, \bar{t}_1) - v(\bar{x}_1, \bar{t}_1) - \frac{\eta}{T - \bar{t}_1} - \delta\bar{t}_1$$

we get a contradiction to the definition of (\bar{x}_1, \bar{t}_1) in (2.15). Whence (2.16) is proved.

We now distinguish two cases: $t_0 = 0$ and $t_0 \in (0, T)$. Assume first that $t_0 = 0$; in order to contradict relation (2.13), observe that

$$\sup_{\tilde{G}} \Psi \leq u(\tilde{x}, \tilde{t}) - v(\tilde{\xi}, \tilde{s}) - \frac{\eta}{T - \tilde{t}} - \delta\tilde{t}.$$

Passing to the limit as $\alpha \rightarrow 0^+$ and taking into account relation (2.14) and $u(x, 0) \leq v(x, 0)$ on Γ , we infer

$$\delta_0/2 \leq u(x_0, 0) - v(x_0, 0) - \eta/T \leq -\eta/T < 0$$

which is the desired contradiction to (2.13).

Assume now that $t_0 > 0$. For α sufficiently small, both \tilde{t} and \tilde{s} are strictly positive. For $x_0 \notin V$ (i.e., $x_0 \in e_j$ for some $j \in J$), one can accomplish the proof by standard arguments so we consider only the case $x_0 = x_i \in V$. We assume without any loss of generality that the unique path of length $d(\tilde{x}, \tilde{\xi})$ connecting \tilde{x} with $\tilde{\xi}$ runs at most through one vertex and, if this happens, the vertex is x_i .

We observe that the functions

$$\begin{aligned} (x, t) &\mapsto u(x, t) - \left[\frac{\eta}{T - t} + \delta t + \frac{d(x, \tilde{\xi})^2}{2\alpha} + \frac{|t - \tilde{s}|^2}{2\alpha} \right] =: u(x, t) - \psi_1(x, t) \\ (\xi, s) &\mapsto v(\xi, s) - \left[-\frac{d(\tilde{x}, \xi)^2}{2\alpha} - \frac{|\tilde{t} - s|^2}{2\alpha} \right] =: v(\xi, s) - \psi_2(\xi, s) \end{aligned}$$

attain respectively a maximum at (\tilde{x}, \tilde{t}) and a minimum at $(\tilde{\xi}, \tilde{s})$. We now distinguish several cases according to the position of \tilde{x} and $\tilde{\xi}$.

Case 1: $\tilde{x} \in e_j$, $\tilde{\xi} \in e_k$ for some $j, k \in \text{Inc}_i$ $j \neq k$. We note that the functions ψ_1 and ψ_2 are admissible test function respectively at (\tilde{x}, \tilde{t}) and at $(\tilde{\xi}, \tilde{s})$; therefore, the definition of sub- and of supersolution entails

$$\frac{\eta}{(T - \tilde{t})^2} + \delta + \frac{\tilde{t} - \tilde{s}}{\alpha} + H\left(\tilde{x}, \frac{d(\tilde{x}, \tilde{\xi})}{\alpha}\right) \leq 0 \quad (2.17)$$

$$\frac{\tilde{t} - \tilde{s}}{\alpha} + H\left(\tilde{\xi}, \frac{d(\tilde{x}, \tilde{\xi})}{\alpha}\right) \geq 0. \quad (2.18)$$

Subtracting the latter inequality from the former, we deduce

$$\delta \leq \frac{\eta}{(T - \tilde{t})^2} + \delta \leq H\left(\tilde{\xi}, \frac{d(\tilde{x}, \tilde{\xi})}{\alpha}\right) - H\left(\tilde{x}, \frac{d(\tilde{x}, \tilde{\xi})}{\alpha}\right).$$

By (2.4), we obtain

$$\left| H\left(\tilde{\xi}, \frac{d(\tilde{x}, \tilde{\xi})}{\alpha}\right) - H\left(\tilde{x}, \frac{d(\tilde{x}, \tilde{\xi})}{\alpha}\right) \right| \leq C d(\tilde{x}, \tilde{\xi}) \left(1 + \frac{d(\tilde{x}, \tilde{\xi})}{\alpha}\right).$$

By the last two inequalities and relation (2.16), passing to the limit as $\alpha \rightarrow 0^+$, we conclude $\delta \leq 0$ which is the desired contradiction.

Case 2: $\tilde{x} \in e_j$, $\tilde{\xi} = x_i$ for some $j \in Inc_i$. As before, the function ψ_1 is an admissible test function at (\tilde{x}, \tilde{t}) . We claim that the function ψ_2 is an admissible (j, k) -test function for every $k \in Inc_i \setminus \{j\}$ (recall that the point \tilde{x} belongs to the edge e_j). Actually, for $k \neq j$, we have

$$D_j \psi_2(\tilde{\xi}, \tilde{s}) = -a_{ij} \frac{d(\tilde{x}, \tilde{\xi})}{\alpha}, \quad D_k \psi_2(\tilde{\xi}, \tilde{s}) = a_{ik} \frac{d(\tilde{x}, \tilde{\xi})}{\alpha};$$

hence, condition (2.8) is satisfied. In particular, ψ_2 is an admissible (j, k) -test function where k is the i -feasible edge for j at $(\tilde{\xi}, \tilde{s})$ in the definition of supersolution. The definition of sub- and of super solution yields again inequalities (2.17) and (2.18). Since the rest of the proof can be achieved following the arguments of the previous case, we shall omit it.

Case 3: $\tilde{x} = x_i$, $\tilde{\xi} \in e_j$ for some $j \in Inc_i$. This case is similar (and even simpler, because the definition of subsolution is less restrictive than the one of supersolution) to Case 2; hence, we shall omit it.

Case 4: $\tilde{x} = x_i = \tilde{\xi}$. Let us just observe that

$$D_j \psi_1(x_i, \tilde{t}) = 0 = D_j \psi_2(x_i, \tilde{s}) \quad \forall j \in Inc_i;$$

namely, ψ_1 (respectively, ψ_2) is an admissible (j, k) -test function at (\tilde{x}, \tilde{t}) (resp., at $(\tilde{\xi}, \tilde{s})$) for every $j, k \in Inc_i$ with $j \neq k$. By the same arguments as those in Case 2 and 3, we infer inequalities (2.17) and (2.18) and we conclude as in Case 1. \square

2.2 A regularity result

This section is devoted to establish a regularity result for solution of the equation (2.1).

Proposition 2.2 *Let u be a solution to (2.1)-(2.2) where u_0 is Lipschitz continuous. Then, u is Lipschitz continuous in $\Gamma \times [0, T]$.*

PROOF We shall follow some ideas of [13, Proposition 2.1]. Let L and M be the Lipschitz constant of u_0 and respectively the maximum of $H(x, p)$ in $\Gamma \times [-L, L]$. We note that the functions

$$(x, t) \mapsto u_0(x) \pm Mt$$

are respectively a super- and a subsolution to problem (2.1)-(2.2). The comparison principle ensures

$$u_0(x) - Mt \leq u(x, t) \leq u_0(x) + Mt;$$

hence u is bounded. Moreover, for every $h > 0$, the functions

$$\underline{w}_h(x, t) := u(x, t + h) - Mh, \quad \overline{w}_h(x, t) := u(x, t - h) + Mh$$

are respectively a sub- and a supersolution to problem (2.1)-(2.2). Invoking again the comparison principle, we infer

$$u(x, t + h) - Mh \leq u(x, t) \leq u(x, t - h) + Mh$$

for every $h > 0$ sufficiently small. For each $(x, t) \in \Gamma \times [0, T)$ and for $h > 0$ small, we deduce

$$\left| \frac{u(x, t + h) - u(x, t)}{h} \right| \leq M.$$

which amounts to the Lipschitz continuity in the variable t . Hence, the function u verifies (in viscosity sense)

$$-M \leq H(x, Du) \leq M.$$

By the coercivity of H and the previous estimate, we get the Lipschitz continuity of u in the variable x (see [2, Prop. II.4.1]). \square

2.3 A representation formula

In this section we exploit the geometric character of the equation to give a representation formula for the solution of (2.1)-(2.2). Given the Hamiltonian H , we define the support function of the sub-level set $\{p \in \mathbb{R} : H(x, p) \leq 1\}$ by

$$s(x, q) = \sup\{pq : H(x, p) \leq 1\}.$$

The function $s : \Gamma \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, convex, positive homogeneous and non negative (see [16]). For example, for the eikonal Hamiltonian $H(x, p) = |p|/c(x)$ (introduced in Example 1), there holds $s(x, q) = c(x)|q|$.

A path $\xi : [0, t] \rightarrow \Gamma$ is said *admissible* if there are $t_0 = 0 < t_1 < \dots < t_{M+1} = t$ such that, for any $m = 0, \dots, M$, $\xi([t_m, t_{m+1}]) \subset e_{j_m}$ for some $j_m \in J$ and $\pi_{j_m}^{-1} \circ \xi \in C^1(t_m, t_{m+1})$. We denote by $B_{y,x}^t$ the set of the admissible path such that $\xi(0) = y$, $\xi(t) = x$.

We introduce a distance function related to the Hamiltonian H on the network. For $x, y \in \Gamma$ define

$$S(y, x) = \inf \left\{ \int_0^t s(\xi(r), \dot{\xi}(r)) dr : t > 0, \xi \in B_{y,x}^t \right\}. \quad (2.19)$$

Note that the distance defined by (2.19) coincides with the one defined by (1.3) for $H(x, p) = |p|$ since in this case $s(x, q) = |q|$. The next proposition summarizes some properties of S (for the definition of viscosity solution on a network in the stationary case, we refer the reader to the paper [14]).

Proposition 2.3 *S is a Lipschitz continuous function on $\Gamma \times \Gamma$ and it is equivalent to the path distance d , i.e. there exists $C > 0$ such that*

$$Cd(x, y) \leq S(x, y) \leq \frac{1}{C} d(x, y), \quad \text{for any } x, y \in \Gamma. \quad (2.20)$$

Moreover, for any $K \subset \Gamma$, closed, $S(K, \cdot)$ is a subsolution in Γ and a supersolution in $\Gamma \setminus K$ of the Hamilton-Jacobi equation

$$H(x, Du) = 1. \quad (2.21)$$

For the proof of the previous proposition we refer to [14, Prop. 4.1].

Theorem 2.2 *Let $u_0 : \Gamma \rightarrow \mathbb{R}$ be a continuous function. Then the solution of (2.1)-(2.2) is given by*

$$u(x, t) = \min\{u_0(y) : S(y, x) \leq t\}. \quad (2.22)$$

In order to prove this result, let us first establish a preliminary lemma.

Lemma 2.1 *If w is a subsolution (resp., supersolution) of (2.21) in Γ , then $u(x, t) = w(x) - t$ is a subsolution (resp., supersolution) of (2.1) in $\Gamma \times (0, T)$.*

PROOF We only show that u is a subsolution at (x_0, t_0) in the case $x_0 = x_i \in V$, being the case $x_0 \notin V$ similar. Assume by contradiction that there exists $\eta > 0$, $j, k \in \text{Inc}_i$ and a (j, k) -admissible test function ϕ at (x_0, t_0) such that $u - \phi$ has a local maximum at (x_0, t_0) relatively to $(e_j \cup e_k) \times (0, T)$ and

$$\partial_t \phi(x_0, t_0) + H(x_0, D\phi(x_0, t_0)) \geq \eta > 0. \quad (2.23)$$

By

$$u(x_0, t_0) - \phi(x_0, t_0) \geq u(x, t) - \phi(x, t) \quad (2.24)$$

for $x = x_0$, taking into account the definition of u , we get $\phi(x_0, t) - \phi(x_0, t_0) \geq t_0 - t$ $\forall t \in (0, T)$; by the arbitrariness of t , we infer

$$\partial_t \phi(x_0, t_0) = -1. \quad (2.25)$$

Moreover, by (2.24) with $t = t_0$, we get that $w(x) - \phi(x, t_0)$ has a local maximum at x_0 relatively to $e_j \cup e_k$, hence

$$H(x_0, D\phi(x_0, t_0)) \leq 1. \quad (2.26)$$

By (2.25) and (2.26) we get a contradiction to (2.23).

The proof that u is a supersolution in $x_0 = x_i \in \Gamma_T$ is similar. We claim that u verifies the supersolution condition choosing as i -feasible edge for $j \in \text{Inc}_i$ the same edge e_k which is i -feasible for j for the function w at x_i . The calculations follows the same arguments as before so we shall omit them. \square

PROOF OF THEOREM 2.2

i) u is continuous: Given $(x_0, t_0) \in \Gamma \times [0, T]$, let $(x_n, t_n) \in \Gamma \times [0, T]$ be such that $\lim_{n \rightarrow \infty} (x_n, t_n) = (x_0, t_0)$ and set $\delta_n = |t_n - t_0| + Cd(x_n, x_0)$ where C is as in (2.20). We claim that

$$\{y \in \Gamma : S(y, x_0) \leq t_0\} \subset \{y : S(y, x_n) \leq t_n + \delta_n\}. \quad (2.27)$$

Indeed, if $S(y, x_0) \leq t_0$, then

$$S(y, x_n) \leq S(y, x_0) + S(x_0, x_n) \leq t_0 + Cd(x_0, x_n) \leq t_n + \delta_n$$

and therefore (2.27). Moreover we claim

$$\{y \in \Gamma : S(y, x_n) \leq t_n + \delta_n\} \subset \{y \in \Gamma : d(y, \{z : S(z, x_n) \leq t_n\}) \leq \delta_n/C\}. \quad (2.28)$$

Indeed, if $S(y, x_n) \leq t_n + \delta_n$, then

$$d(y, \{z : S(z, x_n) \leq t_n\}) \leq \frac{1}{C} S(y, \{z : S(z, x_n) \leq t_n\}) \leq \frac{\delta_n}{C}$$

where the latter inequality is due to the subadditivity of S ; hence, relation (2.28) follows. Therefore, by (2.27) and (2.28), we deduce

$$\begin{aligned} u(x_0, t_0) &= \min\{u_0(y) : S(y, x_0) \leq t_0\} \\ &\geq \min\{u_0(y) : S(y, x_n) \leq t_n + \delta_n\} \\ &\geq \min\{u_0(y) : d(y, \{z : S(z, x_n) \leq t_n\}) \leq (\delta_n + t_n)/C\} \\ &\geq u(x_n, t_n) - \omega(\delta_n/C) \end{aligned}$$

where ω is the modulus of continuity for u_0 in a neighborhood of x_0 . This gives

$$u(x_0, t_0) \geq \limsup_{(x_n, t_n) \rightarrow (x_0, t_0)} u(x_n, t_n).$$

By $\{y \in \Gamma : S(y, x_n) \leq t_n\} \subset \{y \in \Gamma : S(y, x_0) \leq t_0 + \delta_n\}$ we get in a similar way

$$u(x_0, t_0) \leq \liminf_{(x_n, t_n) \rightarrow (x_0, t_0)} u(x_n, t_n).$$

ii) u is a supersolution: We only prove that u is a supersolution at (x_0, t_0) with $x_0 = x_i \in V$. Assume by contradiction that, for some $j \in Inc_i$ and for any $k \in Inc_i \setminus \{j\}$, there is a (j, k) -admissible test function ϕ_k such that $u - \phi_k$ has a local minimum at (x_0, t_0) relatively to $e_j \cup e_k$ with $\phi_k(x_0, t_0) = u(x_0, t_0) = \alpha$ and such that

$$\partial_t \phi_k(x_0, t_0) + H(x, D\phi_k) \leq -\delta < 0. \quad (2.29)$$

Observe that

$$\{(x, t) : u(x, t) \leq \alpha\} = \{(x, t) : S(\{u_0 \leq \alpha\}, x) \leq t\}. \quad (2.30)$$

Assume first that $S(\{u_0 \leq \alpha\}, x_0) > 0$ and define $w(x, t) = S(\{u_0 \leq \alpha\}, x) - t$. We claim that ϕ_k has a local maximum on the set $\{w \leq 0\} \cap ((e_j \cup e_k) \times (0, T))$ at (x_0, t_0) . In fact if $(x, t) \in \{w \leq 0\}$ then by (2.30), $u(x, t) \leq \alpha$ and since

$$0 = u(x_0, t_0) - \phi_k(x_0, t_0) \leq u(x, t) - \phi_k(x, t) \quad (2.31)$$

we get $\phi_k(x, t) \leq u(x, t) \leq \alpha \leq \phi(x_0, t_0)$ and the claim is proved. By Lemma 2.1 w is supersolution to (2.1) and therefore Prop.2.1 gives a contradiction to (2.29).

If $S(\{u_0 \leq \alpha\}, x_0) = 0$, we claim that (x_0, t_0) is a local maximum point for u . In fact, $S(\{u_0 \leq \alpha\}, x_0) = 0 \leq t_0 - \eta$ for some $\eta > 0$. If (x, t) is such that $\max\{S(x, x_0), |t - t_0|\} \leq \delta/2$ with $\delta < \eta$, then

$$S(\{u_0 \leq \alpha\}, x) \leq S(\{u_0 \leq \alpha\}, x_0) + S(x_0, x) \leq t_0 - \eta + \delta/2 \leq t$$

hence $u(x, t) \leq \alpha = u(x_0, t_0)$ and the claim is proved. By (2.31) (x_0, t_0) is also a local maximum point for ϕ_k on $(e_j \cup e_k) \times (0, T)$ for any $k \neq j$. By (2.7) and (2.29), we get $\phi_t(x_0, t_0) < 0$ and therefore a contradiction to (x_0, t_0) being a local maximum point for ϕ_k .

iii) u is a subsolution: We only consider the case (x_0, t_0) with $x_0 = x_i \in V$. We first observe that

$$\overline{\{(x, t) : u(x, t) > \alpha\}} = \{(x, t) : S(\{u_0 \leq \alpha\}, x) \geq t\}.$$

Given $j, k \in Inc_i$ let ϕ be an (j, k) -test function such that $u - \phi$ has a local maximum at (x_0, t_0) relatively to $(e_j \cup e_k) \times (0, T)$. Arguing as in the supersolution case we define $w(x, t) = S(\{u_0 \leq \alpha\}, x) - t$ and we show that ϕ has a local minimum on the set $\{w \geq 0\} \cap ((e_j \cup e_k) \times (0, T))$. Then we conclude by applying Prop.2.1 and Lemma 2.1. \square

3 An application: the blocking problem

In this section we provide a concrete application of our results: now, the network Γ represents an oil pipeline (a network of computer, the circulatory system, etc.) and at initial time a fire breaks up in the region $R_0 \subset \Gamma$ (a virus is detected in a subnet, an embolus occurs in some vessel). The speed of propagation of the fire is known but it may depend on the state variable. Our aim is to determine an optimal strategy to stop the fire and to minimize the burnt region.

As in the flame propagation model described in [3], let R_0 be the initial burnt region and R_t the region burnt at time t . Assume that the front ∂R_t propagates in the outward normal direction to the front itself. Then R_t is given by the 0-sublevel set of a viscosity solution of (2.1) -(2.2) where the initial datum u_0 satisfies $R_0 = \{x \in \Gamma : u_0(x) \leq 0\}$.

Recalling the representation formula (2.22) we observe that the 0-sublevel set of the solution of (2.1)-(2.2) is given by

$$R_t = \{x \in \Gamma : S(R_0, x) \leq t\}$$

where S is defined as (2.19). Note that, since Γ is composed by a finite number of bounded edges and therefore its total length is finite, then the burnt region

$$R = \{x \in \Gamma : S(R_0, x) < \infty\} = \cup_{t \geq 0} R_t$$

coincides with Γ . In other words, without any external intervention, the pipeline will be completely burnt in a finite time.

We assume that an operator, located at $x_0 \in V$ (the “operation center”), can block the fire by closing the junctions of the pipeline (i.e., vertices of the network) and that this operation is effective only after a delay which depends on the distance of the junction from x_0 .

Our problem is reminiscent of other models described in literature (for instance, see [10, 18] and references therein) which concern the control of some diffusion in a network (e.g. minimizing the spread of a virus or maximizing the spread of an information). In this framework, let us stress the main novelties of our setting: in our model, the diffusion has positive finite speed and it affects both vertices and edges, the spread is not reversible (namely, “infected” points cannot become again “healthy”) and the effect of the operator’s action has finite speed (in other words, it is effective after a delay depending on the distance from the operation center).

Definition 3.1 *An admissible strategy σ is a subset of V such that*

$$S(R_0, x_i) \geq \delta d(x_0, x_i) \quad \forall x_i \in \sigma \quad (3.1)$$

where δ is a given nonnegative constant. We denote by V_{ad} the vertices of the network which satisfy the admissibility condition (3.1) and by Σ_{ad} the set of the admissible strategies.

Remark 3.1 *Condition (3.1) means that the time to reach the vertex $x_i \in \sigma$ from x_0 at the velocity $1/\delta$ is less than or equal to the time the fire front reaches x_i . Therefore the junction x_i can be blocked before the front goes through it.*

Given a strategy $\sigma \in \Sigma_{ad}$, we denote $S^\sigma : \Gamma \times \Gamma \rightarrow [0, \infty]$ the distance restricted to the trajectories not going through a vertex in σ , i.e.

$$S^\sigma(y, x) := \inf \left\{ \int_0^t s(\xi(r), \dot{\xi}(r)) dr : t > 0, \xi \in B_{y,x}^t \text{ s.t. } \xi(r) \notin \sigma \ \forall r \in [0, t] \right\}$$

with $S^\sigma(y, x) = \infty$ if there is no admissible curve joining y to x . We also set

$$\begin{aligned} R_t^\sigma &:= \{x \in \Gamma : S^\sigma(R_0, x) \leq t\} \\ R^\sigma &:= \cup_{t \geq 0} R_t^\sigma = \{x \in \Gamma : S^\sigma(R_0, x) < \infty\} \end{aligned}$$

which are respectively the region burnt at time t and the total burnt region using the strategy σ . Observe that

- if δ is very small, then the optimal strategy is given by the extremes of the edges containing R_0 ;

- if δ is very large and $x_0 \in R_0$, then every strategy is useless since the whole pipeline will burn whatever the operator does.

Aside the previous simple cases an optimal strategy for the blocking problem may be not obvious and we aim to find an efficient way to compute it. To find a strategy which minimizes the burnt region, we first give a characterization of R^σ in terms of a problem satisfied by the distance $S^\sigma(R_0, \cdot)$.

Proposition 3.1 *Given $\sigma \in \Sigma_{ad}$, set $u(x) = S^\sigma(R_0, x)$ and $\mathcal{R} = R^\sigma$. Then*

- i) $u \in C^0(\mathcal{R})$ and $u = +\infty$ in $\Gamma \setminus \mathcal{R}$. Moreover if $x_i \in \sigma$ and $j \in \text{Inc}_i$ is such that $e_j \subset \mathcal{R}$, then $\lim_{x \rightarrow x_i, x \in e_j} u(x) = u_j(x_i) < \infty$;*
- ii) u is a viscosity solution of the problem*

$$\begin{cases} H(x, Du) = 1, & x \in \mathcal{R} \setminus (R_0 \cup \sigma) \\ u = 0, & x \in R_0; \end{cases} \quad (3.2)$$

- iii) let $w \in USC(\Gamma)$ be such that, defined $\mathcal{R}_w = \{x \in \Gamma : w(x) < \infty\}$,*

$$\begin{cases} H(x, Dw) \leq 1, & x \in \mathcal{R}_w \setminus R_0 \\ w = 0, & x \in R_0, \end{cases}$$

then $\mathcal{R} \subset \mathcal{R}_w$ and $w \leq u$ in Γ .

PROOF Note that, for $j \in J$, either $e_j \subset \mathcal{R}$ or $e_j \cap \mathcal{R} = \emptyset$, i.e. an edge is either completely burnt or it cannot be reached by the fire. The function u can be discontinuous at $x_i \in \sigma$ and

- if $x_i \in V \setminus \sigma$, then either $u_j(x_i) = \infty$ for all $j \in \text{Inc}_i$ if $x_i \in \Gamma \setminus \mathcal{R}$ or $u_j(x_i) < \infty$ for all $j \in \text{Inc}_i$ if $x_i \in \mathcal{R}$;
- if $x_i \in \sigma$, then either $u_j(x_i) = \infty$ for all $j \in \text{Inc}_i$ if $x_i \in \Gamma \setminus \mathcal{R}$ or there exists $j \in \text{Inc}_i$ such that $u_j(x_i) < \infty$ if $x_i \in \mathcal{R}$ and in this case $u_j(x_i) = \sup_{e_j} u_j$.

Actually, if $x_i \in \sigma$ and $u_j(x_i) < \infty$, an admissible trajectory for S^σ connecting x_i to R_0 and containing the edge e_j , $j \in \text{Inc}_i$, necessarily enters from x_i into e_j . Hence $u(x)$ is increasing for $x \in e_j$, $x \rightarrow x_i$ and $\lim_{x \in e_j, x \rightarrow x_i} u_j(x) = u_j(x_i)$.

In $\mathcal{R} \setminus \sigma$, S^σ locally behaves as the distance S defined in (2.19). Therefore the continuity of u in $\mathcal{R} \setminus \sigma$ and the sub and supersolution properties in the open set $\mathcal{R} \setminus (R_0 \cup \sigma)$ are obtained with the same arguments [14, Prop.4.1].

To prove *iii)*, assume by contradiction that there exists $x_0 \in \mathcal{R}$ such that $u(x_0) < w(x_0)$. For any $x, y \in \mathcal{R}$ such that $S^\sigma(x, y) < \infty$, a minimizing trajectory always exists since (up to reparametrization) there is only a finite number of trajectories connecting the two points.

Hence let ξ be an admissible curve for S^σ such that $\xi(0) = y_0 \in R_0$, $\xi(T) = x_0$ and $u(x_0) = \int_0^T s(\xi(r), \dot{\xi}(r)) ds$. Let $t_0 = 0 < t_1 < \dots < t_{M+1} = T$ such that, for

any $m = 0, \dots, M$, $\xi([t_m, t_{m+1}]) \subset e_{j_m}$ for some $j_m \in J$, $\xi(t_{i_m}) = x_{i_m} \in V$ and $\pi_{j_m}^{-1} \circ \xi \in C^1(t_m, t_{m+1})$. Clearly $u(\xi(t)) = \int_0^t s(\xi(r), \dot{\xi}(r)) dr$ for $t \in [0, T]$.

If w is a subsolution to (3.2), then by the coercivity of H , w is Lipschitz continuous in $\mathcal{R}_w \setminus R_0$ and therefore $H(x, Dw) \leq 1$ a.e. on $\mathcal{R}_w \setminus R_0$. Moreover, by the definition of the support function s , we have $H(x, p) \leq 1$ if and only if $\sup_{q \in \mathbb{R}} \{pq - s(x, q)\} \leq 0$. Hence

$$a_{i_m j_m} \dot{\xi}(r) Dw(\xi(r)) \leq s(\xi(r), \dot{\xi}(r)) \quad \forall r \in [t_m, t_{m+1}], m = 0, \dots, M.$$

(the term $a_{i_m j_m}$ takes into account the orientation of the arc e_{j_m}). Integrating the previous relation in $[0, t_1]$ and recalling that $u(y_0) = w(y_0) = 0$, we get

$$w(\xi(t_1)) \leq \int_0^{t_1} s(\xi(r), \dot{\xi}(r)) dr = u(\xi(t_1)).$$

Iterating the same argument in $[t_m, t_{m+1}]$ we finally get $w(\xi(T)) \leq u(\xi(T))$ and therefore a contradiction since $\xi(T) = x_0$. We conclude that $x_0 \in \mathcal{R}_w$ and $w(x_0) \leq u(x_0)$. \square

We now show that the strategy composed by all the admissible nodes which are adjacent to a non admissible node is optimal, in the sense that it maximizes the preserved region.

Proposition 3.2 *The admissible strategy*

$$\sigma_{opt} = \{x_i \in V_{ad} : \exists x_j \in V \setminus V_{ad}, e_k \in E \text{ s.t. } x_i, x_j \in e_k\}$$

satisfies: for any $\sigma \in \Sigma_{ad}$, $\Gamma \setminus R^\sigma \subset \Gamma \setminus R^{\sigma_{opt}}$.

PROOF Assume by contradiction that there exist $\sigma \in \Gamma_{ad}$ and $x_0 \in (\Gamma \setminus R^\sigma) \cap R^{\sigma_{opt}}$. Hence there exists an admissible trajectory ξ for σ_{opt} connecting x_0 to R_0 , i.e. there exists $y_0 \in R_0$, $t > 0$ and $\xi \in B_{y_0, x_0}^t$ such that $\xi(r) \notin \sigma_{opt}$ for $r \in [0, t]$. Note that σ_{opt} disconnect the subgraph containing the admissible vertices V_{ad} by the one containing the non admissible vertices $V \setminus V_{ad}$ and therefore $\xi([0, t])$ is contained in the subgraph with vertices $(V \setminus V_{ad}) \cup \sigma_{opt}$. Since $\sigma \subset V_{ad}$, then ξ is also admissible for $S^\sigma(y_0, x_0)$ and therefore a contradiction to $x_0 \in \Gamma \setminus R^\sigma$. \square

Remark 3.2 *One could be interested in a cost functional on Σ_{ad} which takes into account other terms, as the cost of blocking a junction, beside the length of the burnt region. For instance, consider the cost*

$$\mathcal{I}(\sigma) = \sum_{x_i \in \sigma} \alpha_i + \sum_{e_j \in R^\sigma} \beta_j \quad \sigma \in \Sigma_{ad}$$

where α_i represents the cost of blocking the vertex x_i while β_j represents the damage of the destruction of edge e_j by fire. Clearly, the minimum of \mathcal{I} exists since Σ_{ad} is finite, but it seems more difficult to characterize the optimal strategy because it will strongly depend on the geometry of the network.

3.1 Numerical simulations

In this section we propose a numerical method to compute the optimal strategy for the blocking problem. The scheme is based on a finite difference approximation of the stationary problem (3.2); for simplicity, we only consider the case of the eikonal Hamiltonian $H(x, p) = |p|/c(x)$.

On each interval $[0, l_j]$ parametrizing the arc e_j , we consider an uniform partition $y_{j,m} = mh_j$ with $M_j = l_j/h_j \in \mathbb{N}$ and $m = 0, \dots, M_j$. In this way we obtain a grid $\mathcal{G}^h = \{x_{j,m} = \pi_j(y_{j,m}), j \in J, m = 0, \dots, M_j\}$ on the network Γ . We define $\mathcal{R}_0^h = \mathcal{G}^h \cap R_0$, the set of the nodes in the initial front. For $x_1, x_2 \in \mathcal{G}^h$, we say that x_1 and x_2 are adjacent and we write $x_1 \sim x_2$ if and only if they are the image of two adjacent grid points, i.e. $x_k = \pi_j(y_k)$, for $y_k \in [0, l_j]$, $k = 1, 2$, $j \in J$ and $|y_1 - y_2| = h_j$. Note that if $x_i \in V$ is a vertex of Γ , then the nodes of the grid \mathcal{G}^h adjacent to x_i may belong to different arcs.

We compute the optimal strategy by means of the following Algorithm, based on the results of Prop. 3.2.

Blocking strategy [B]

- (i) In the first step we solve the front propagation problem on the network computing the approximated time $u^h(x)$ at which a node $x_{j,m} \in \mathcal{G}^h$ gets burnt

$$\begin{cases} \max_{x \in \mathcal{G}^h, x \sim x_{j,m}} \left\{ -\frac{1}{h_j}(u^h(x) - u^h(x_{j,m})) \right\} - c(x_{j,m}) = 0 & x_{j,m} \in \mathcal{G}^h \\ u^h(x_{j,m}) = 0 & x_{j,m} \in \mathcal{R}_0^h \end{cases} \quad (3.3)$$

Note that if $x_{j,m}$ coincides with a vertex $x_i \in V$, the approximating equation reads as

$$\max_{j \in Inc_i} \max_{x \in \mathcal{G}^h \cap e_j, x \sim x_{j,m}} \left\{ -\frac{1}{h_j}(u^h(x) - u^h(x_{j,m})) \right\} - c(x_{j,m}) = 0.$$

The discrete function $u^h : \mathcal{G}^h \rightarrow \mathbb{R}$ is such that $u^h(x_{j,m}) \simeq u(x_{j,m})$, where $u(x) = S(R_0, x)$.

- (ii) In the second step we determine the vertices which satisfy the admissibility condition (3.1). We define $V_{ad}^h = \{x_i \in V : w^h(x_i) < u^h(x_i)\}$, where $w^h : \mathcal{G}^h \rightarrow \mathbb{R}$ represents the approximated time to reach a node $x \in \mathcal{G}^h$, starting from the operation center x_0 and moving with a constant speed $1/\delta$. The function w^h is computed by means of the finite difference scheme

$$\begin{cases} \max_{x \in \mathcal{G}^h, x \sim x_{j,m}} \left\{ -\frac{1}{h_j}(w^h(x) - w^h(x_{j,m})) \right\} - \frac{1}{\delta} = 0 & x_{j,m} \in \mathcal{G}^h \setminus \{x_0\} \\ w^h(x_0) = 0. \end{cases} \quad (3.4)$$

- (iii) We define the approximated optimal strategy by setting

$$\sigma_{opt}^h = \{x_i \in V_{ad}^h : \exists x_j \in V \setminus V_{ad}^h, e_k \in E \text{ s.t. } x_i, x_j \in e_k\}$$

and, for $\sigma = \sigma_{opt}^h$, we compute the corresponding approximate distance by solving the following finite difference scheme

$$\begin{cases} \max_{x \in \mathcal{G}^h, x \sim x_{j,m}} \left\{ -\frac{1}{h_j} (u_\sigma^h(x) - u_\sigma^h(x_{j,m})) \right\} - c(x_{j,m}) = 0 & x_{j,m} \in \mathcal{G}^h \setminus (\mathcal{R}_0^h \cup \sigma) \\ u_\sigma^h(x_{j,m}) = 0 & x_{j,m} \in \mathcal{R}_0^h \\ -\frac{1}{h_j} (u_{j,\sigma}^h(x) - u_{j,\sigma}^h(x_{j,m})) - c(x_{j,m}) = 0 & x_{j,m} \in \sigma, x \in \mathcal{G}^h, x \sim x_{j,m} \end{cases}$$

The discrete function $u^h : \mathcal{G}^h \rightarrow \mathbb{R}$ is such that $u_\sigma^h(x_{j,m}) \simeq u(x_{j,m})$, where $u(x) = S^\sigma(R_0, x)$ solves (3.2). Note that as in the continuous case, the value of u^h at $x_i \in \sigma$ can depend on the edge e_j and in general the function is discontinuous at these points.

Remark 3.3 *In this paper we do not analyze the properties of the previous finite difference schemes. In any case, at least for (3.3) and (3.4), the well-posedness and the convergence of the schemes can be studied by adapting the techniques in [5].*

3.2 Example 1: a simple network

We consider a network with a simple structure where the fire starts in one vertex, $R_0 = \{(0, 0)\}$ and propagates with speed $c = 1$.

We first perform step *i*) of Algorithm [B] and we compute the approximated time $u^h(x)$ at which a node $x \in \mathcal{G}^h$ get burnt. The results are shown in Fig.1 together with the graph structure.

Next, we perform step *ii*) of Algorithm [B]. We suppose the operation center x_0 is located on the vertex $(-1.5, 2.5)$ and the velocity to reach a node x_i from x_0 is $\frac{1}{\delta} = 1$. Using (3.4), we compute the set of nodes V_{ad}^h . The result is shown in Fig.2, the set of nodes in V_{ad}^h are represented by the square markers.

Once computed the set of admissible nodes V_{ad}^h , we can compute the optimal strategy, following step *iii*). The result is shown in Fig.3. It is clear, from the simple structure of the network, that any other choice of σ_{opt}^h would lead to a greater burnt region and consequently to a smaller preserved network region.

3.3 Example 2: a more complex network

We consider a more complex network, with 20 vertices and 32 arcs. We suppose the fire starts in two vertices and propagates with a non constant normal speed $c(x) = |x|$.

We proceed as in the first example and we compute the approximated time $u^h(x)$ at which a node $x \in \mathcal{G}^h$ get burnt. The results are shown in Fig.4 together with the graph structure.

We suppose the operation center x_0 is located on the vertex $x_0 = (3.8, 6.5)$ and the velocity to reach a node x_i from x_0 is $\frac{1}{\delta} = 1/5$. Using (3.4), we compute the set of nodes V_{ad}^h . The result is shown in Fig. 5, the set of nodes in V_{ad}^h are represented by the square markers. Once computed the set of admissible nodes V_{ad}^h , we can

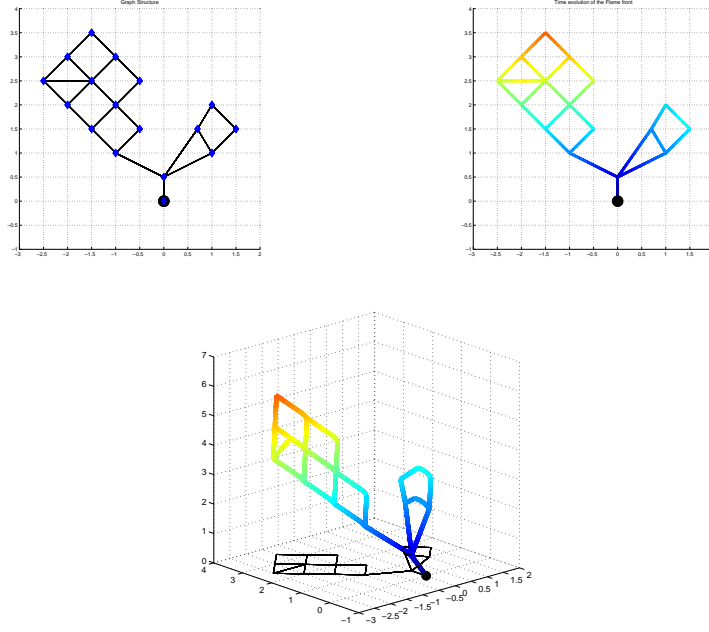


Figure 1: Test1. Graph structure where R_0 is represented by the circle marker and the vertices by the rhombus markers (Top Left). Color map of the time $u_h(x)$ at which a node x get burnt, computed by (3.3), (Top Right) and its 3D view (Bottom).

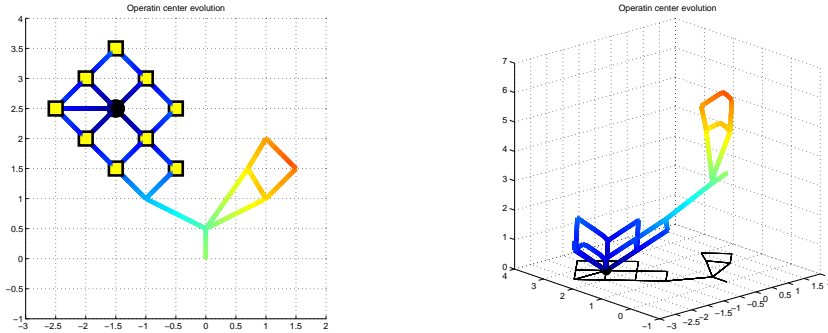


Figure 2: Test1. Time to reach a point x from the operation center x_0 (circle marker) and set of the admissible nodes V_{ad}^h (square marker). 2D view (Left) and 3D view (Right).

compute the optimal strategy, following step *iii*). The result is shown in Fig.6. In this case we get the optimal solution blocking only three vertices. By changing the set R_0 as in Fig.7, the region of the admissible node V_{ad}^h , shown in Fig. 8, turns out to be much smaller. In this case the optimal strategy is formed by all the vertices in V_{ad}^h , and the preserved region becomes smaller than the previous case, see Fig. 9.

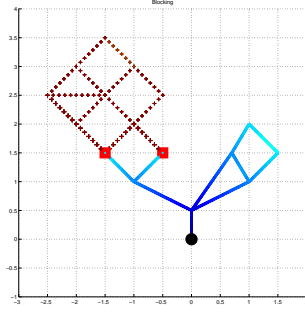


Figure 3: Test1. Optimal blocking strategy σ_{opt}^h (square marker), preserved network region (cross marker) and minimum burnt network region (continuum line) starting from R_0 (circle marker).

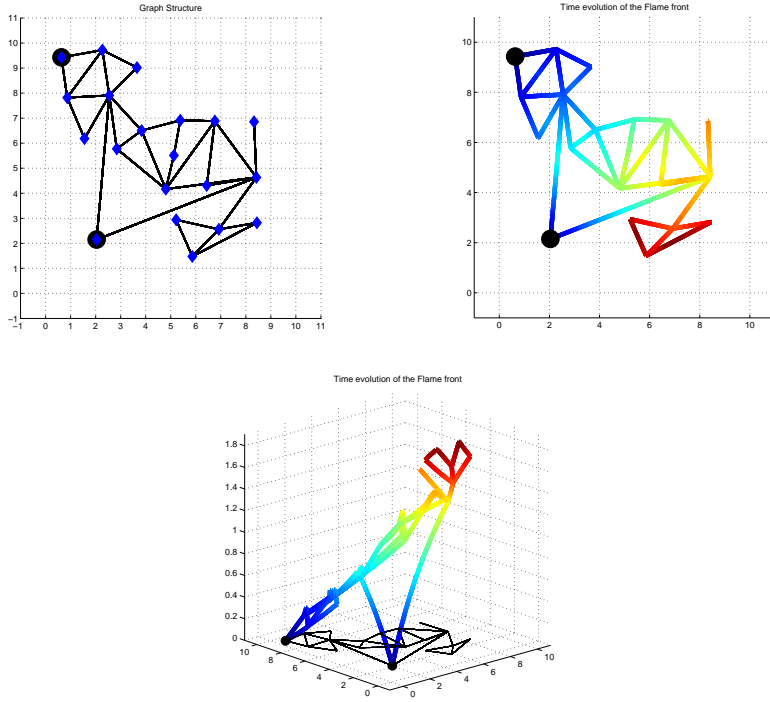


Figure 4: Test2. Graph structure where R_0 is represented by the circle markers and the vertices by the rhombus markers (Top Left). Color map of the time $u_h(x)$ at which a node x get burnt, computed by (3.3) (Top Right), and its 3D view (Bottom).

A Appendix

This section is devoted to the proof of Proposition 2.1. To this end, it is expedient to establish the following result.

Lemma A.1 *Let $u \in USC(\Gamma \times (0, T))$ (respectively, $u \in LSC(\Gamma \times (0, T))$) be a*

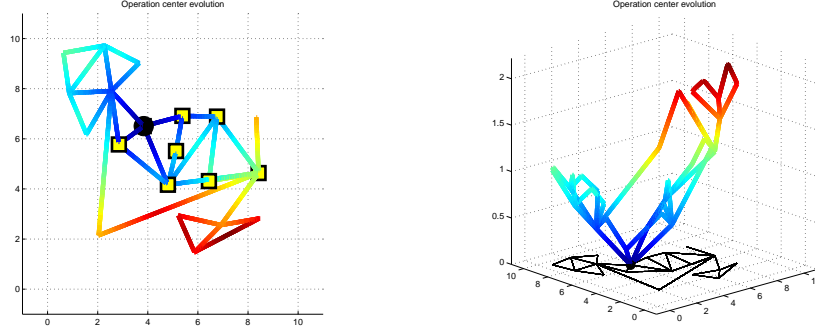


Figure 5: Test2. Time to reach a point x from the operation center x_0 (circle marker) and set of the admissible nodes V_{ad}^h (square markers). 2D view(Left) and 3D view (Right)).

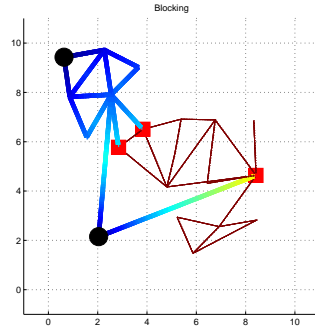


Figure 6: Test2. Optimal blocking strategy σ_{opt}^h (square markers), preserved network region (thin line) and minimum burnt network region (thick line) starting from R_0 (circle markers).

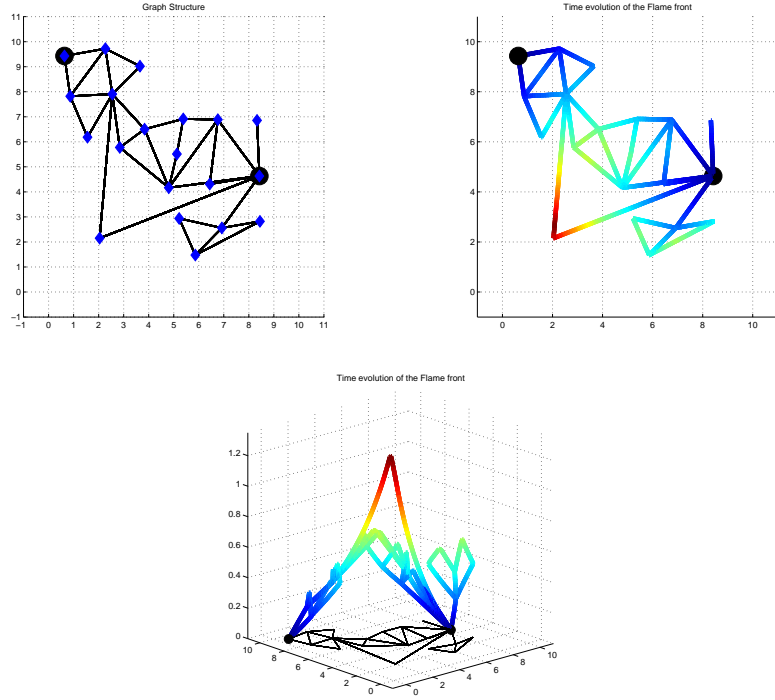


Figure 7: Test3. Graph structure where R_0 is represented by the circle markers and the vertices by the rhombus markers (Top Left). Color map of the time $u_h(x)$ at which a node x get burnt, computed by (3.3), (Top Right) and its 3D view (Bottom).

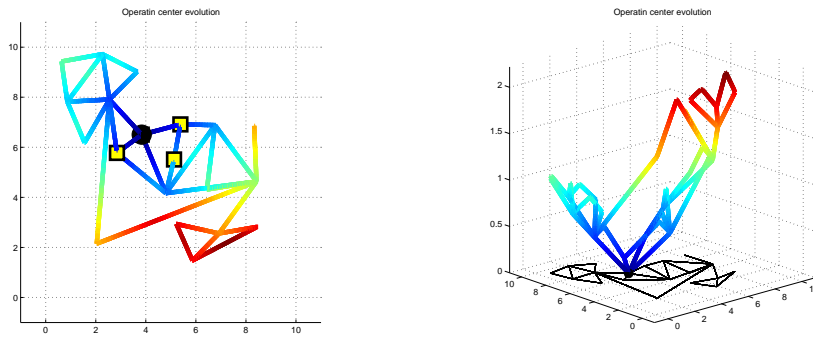


Figure 8: Test3. Time to reach a point x from the operation center x_0 (circle marker) and set of the admissible nodes V_{ad}^h (square markers). 2D view(Left) and 3D view (Right).

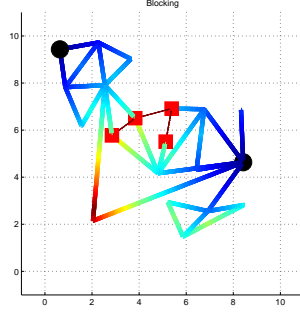


Figure 9: Test3. Optimal blocking strategy σ_{opt}^h (square marker), preserved network region (thin line) and minimum burnt network region (thick line) starting from R_0 (circle marker).

subsolution (resp., supersolution) to (2.1). For any function $\theta : \mathbb{R} \rightarrow \mathbb{R}$ continuous and nondecreasing, the function $\theta \circ u$ is still a subsolution (resp., supersolution) to (2.1).

In particular, if u solves (2.1), then also $\theta \circ u$ solves the same equation.

PROOF For a point inside an edge, the proof follows by the same arguments as in the proof of [7, Theorem 5.2]. For the subsolution case in a vertex x_i , we observe that an USC function is a subsolution if, and only if, it is a standard viscosity subsolution on the arc $e_j \cup e_k$ for any $j, k \in Inc_i$. Taking into account this observation and using again the arguments of [7, Theorem 5.2] we obtain the statement. Finally, the supersolution case at a vertex x_i follows by similar arguments. \square

PROOF OF PROP. 2.1 We shall use the arguments of [15, Lemma 3.1] adapted to the networks.

(i). In this case, in a neighborhood of the point x the network Γ is equivalent to a segment. Therefore, the statement immediately follows from [15, Lemma 3.1].

(ii). Consider a function $u \in USC(\Gamma \times (0, T))$. For $x = x_i \in V$, assume that, for each $\alpha \in \mathbb{R}$, $j, k \in Inc_i$ with $j \neq k$, (j, k) -test function ϕ which has a local minimum on $\{u \geq \alpha\} \cap ((e_j \cup e_k) \times (0, T))$ at (x, t) , inequality (2.9) holds. Our aim is to prove that u satisfies the subsolution condition at (x, t) . To this end, we assume by contradiction that there exists an admissible test function ϕ which verifies

$$0 = u(x, t) - \phi(x, t) > u(y, s) - \phi(y, s) \quad \forall (y, s) \in B_r(x, t) \cap ((e_j \cup e_k) \times (0, T)) \quad (\text{A.1})$$

$$\phi_t(x, t) + H(x, D\phi(x, t)) > 0. \quad (\text{A.2})$$

For $\alpha := \phi(x, t)$, we set $\Delta := \{u \geq \alpha\} \cap ((e_j \cup e_k) \times (0, T))$. Inequality (A.1) and the definition of Δ yield

$$\phi(y, s) \geq u(y, s) \geq \alpha = \phi(x, t) \quad \forall (y, s) \in \Delta \cap B_r(x, t);$$

namely, the function ϕ attains a local minimum in (x, t) with respect to Δ . Invoking our assumption, we infer inequality (2.9) which amounts to the desired contradiction. Hence, the first implication of the statement is achieved.

We now prove the reverse implication. Let u be a subsolution to (2.1). Fix $\alpha \in \mathbb{R}$, $x = x_i \in \Gamma_T$ and $j, k \in \text{Inc}_i$ with $j \neq k$. Let ϕ be an admissible (j, k) -test function which attains a local minimum on $\Delta := \{u \geq \alpha\} \cap ((e_j \cup e_k) \times (0, T))$ at (x, t) .

By Lemma A.1, it suffices to prove that there exists a continuous nondecreasing function θ such that $\theta \circ u - \phi$ attains a local maximum in (x, t) relatively to $(e_j \cup e_k) \times (0, T)$.

Let $W \subset (e_j \cup e_k) \times (0, T)$ be a compact neighborhood of (x, t) in the topology of $(e_j \cup e_k) \times (0, T)$ such that $\phi \geq \phi(x, t)$ in $W \cap \Delta$. We set

$$E_0 := \{(y, s) \in W \mid \phi(y, s) < \phi(x, t)\}$$

(observe that the definition of W ensures: $E_0 \subset (e_j \cup e_k) \times (0, T)$ and $E_0 \cap \Delta = \emptyset$).

We now define the function θ according to the following cases: (a), $E_0 = \emptyset$; (b), $E_0 \neq \emptyset$ and $\beta := \sup_{E_0} u < \alpha$; (c), $E_0 \neq \emptyset$ and $\beta = \alpha$.

Case-(a). For $\theta(s) := \phi(x, t)$ (a constant function), one can easily check that $\theta \circ u - \phi$ attains a local maximum in (x, t) relatively to $(e_j \cup e_k) \times (0, T)$.

Case-(b). We define

$$\theta(s) := \begin{cases} \phi(x, t) & \text{for } s \in [\alpha, +\infty) \\ \inf_W \phi & \text{for } s \in (-\infty, \beta] \\ (\alpha - \beta)^{-1} [(\alpha - s) \inf_W \phi + (s - \beta) \phi(x, t)] & \text{for } s \in (\beta, \alpha). \end{cases}$$

Now, we want to prove

$$\theta(u(y, s)) - \phi(y, s) \leq \theta(u(x, t)) - \phi(x, t) = 0 \quad (\text{A.3})$$

for every $(y, s) \in (e_j \cup e_k) \times (0, T)$ in some neighborhood of (x, t) . To this end, we shall consider separately the cases when (y, s) belongs to $W \setminus (E_0 \cup \Delta)$, E_0 , $W \cap \Delta$. For $(y, s) \in W \setminus (E_0 \cup \Delta)$, by the monotonicity of θ and the definition of Δ and E_0 , we have

$$\theta(u(y, s)) \leq \theta(\alpha) = \phi(x, t) \leq \phi(y, s)$$

which amounts to inequality (A.3). For $(y, s) \in E_0$, taking into account the definition of β , we have

$$\theta(u(y, s)) \leq \theta(\beta) = \inf_W \phi \leq \phi(x, t).$$

For $(y, s) \in W \cap \Delta$, by the definition of Δ (recall that ϕ attains a local minimum on Δ), there holds

$$\theta(u(y, s)) = \phi(x, t) \leq \phi(y, s);$$

hence our claim (A.3) is completely proved.

Case-(c). For any $n \in \mathbb{N}$, we introduce

$$E_n := \{(y, s) \mid \phi(y, s) \leq \phi(x, t) - 1/n\}, \quad \beta_n := \begin{cases} \sup_{E_n} u & \text{if } E_n \neq \emptyset \\ +\infty & \text{if } E_n = \emptyset; \end{cases}$$

we observe that the sets E_n are not empty for n sufficiently large with $E_n \subset E_{n+1}$ and $E_0 = \cup_n E_n$. Moreover, the sequence $\{\beta_n\}$ is nondecreasing and, by the compactness of E_n , it fulfills: $\beta_n < \alpha$ and $\beta_n \rightarrow \alpha$ as $n \rightarrow +\infty$. Hence, there exists an increasing sequence $\{n_m\}$ such that β_{n_m} is increasing and $E_{n_{m+1}} \setminus E_{n_m} \neq \emptyset$. We set

$$\theta(s) := \begin{cases} \phi(x, t) & \text{for } s \in [\alpha, +\infty) \\ \phi(x, t) - 1/n_{m-1} & \text{for } s = \beta_{n_m} \\ \inf_W \phi & \text{for } s \in (-\infty, \beta_{n_1}] \\ \text{linear function} & \text{for } s \in (\beta_{n_{m-1}} - \beta_{n_m}). \end{cases}$$

We want to prove (A.3) studying separately the cases when (y, s) belongs to $W \setminus (E_0 \cup \Delta)$, E_{n_1} , $E_{n_{m+1}} \setminus E_{n_m}$ ($m \in \mathbb{N}$) and $W \cap \Delta$. In the first and last cases, inequality (A.3) follows by the same arguments of point (b). For $(t, s) \in E_{n_1}$ there holds $u(t, s) \leq \beta_{n_1}$ and, by the monotonicity of θ ,

$$\theta(u(y, s)) \leq \theta(\beta_{n_1}) = \inf_W \phi \leq \phi(y, s).$$

For $(y, s) \in E_{n_{m+1}} \setminus E_{n_m}$, there holds $u(y, s) \leq \beta_{n_{m+1}}$ and $\phi(y, s) > \phi(x, t) - 1/n_m$; therefore, we infer

$$\theta(u(y, s)) \leq \theta(\beta_{n_{m+1}}) = \phi(x, t) - 1/n_m < \phi(y, s).$$

Whence, inequality (A.3) is established and statement (ii) is completely proved.

(iii). The proof of this part of the statement follows by the same arguments as those in the previous one reversing each inequality and choosing the test functions only along the i -feasible edge.

□

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